# Thermal Instability of a Compressible and Partially Ionized Plasma

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The compressibility and collisional effects on thermal instability of a composite medium are considered. The effect of compressibility is found to be stabilizing. In contrast to the non-oscillatory modes for $(C_p/g)\,\beta > 1$ in the absence of a magnetic field, $C_p$, $\beta$ and $g$ being the specific heat at constant pressure, a uniform adverse temperature gradient and the acceleration due to gravity respectively, the presence of a magnetic field introduces oscillatory modes in the system. The sufficient condition for non-existence of overstability is found. The magnetic field is found to have a stabilizing effect on the system for $(C_p/g)\,\beta > 1$.

## 1. Introduction

A detailed account on the onset of Bénard convection in incompressible fluids has been given by Chandrasekhar [1]. The Boussinesq approximation has been used throughout, which states that the density may be treated as a constant in all the terms in the equations of motion except the term in the external force. The approximation is well justified in the case of incompressible fluids. When the fluids are compressible, the equations governing the system become quite complicated. To simplify the set of equations governing the flow of compressible fluids, Spiegel and Veronis [2] have made the following assumptions:

(i) The depth of the fluid layer is much smaller than the scale height as defined by Spiegel and Veronis [2] and

(ii) the fluctuations in pressure, density and temperature, introduced by the motion, do not exceed their total static variations.

Under the above assumptions, the flow equations are the same as those for incompressible fluids except that the static temperature gradient is replaced by its excess over the adiabatic.

Quite frequently it happens that the medium is not fully ionized. Following Hans [3], the medium has been idealized as a mixture of a hydromagnetic (ionized) component and a neutral component, the two interacting through mutual collisional (frictional) effects. Hans [3] found that these collisions stabilize the Rayleigh-Taylor instability. For the case of two uniform superposed media in relative horizontal motion (the Kelvin-Helmholtz configuration), Rao and Kalra [4] and Hans [3] found that the collisional effects are in fact destabilizing for a sufficiently large collision frequency. The thermal hydromagnetic instability of a partially ionized medium has been studied by Sharma [5].

The object of the present paper is to study the compressibility and collisional effects on the thermal instability of a partially ionized plasma.

## 2. The Physical Problem, Formulation and Dispersion Relation

Consider an infinite, horizontal, compressible, and composite layer of thickness $d$ consisting of a conducting hydromagnetic fluid of density $\varrho$, permeated with neutrals of density $\varrho_d$, acted on by a gravity force $\boldsymbol{g}(0, 0, -g)$ and a uniform horizontal

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magnetic field $\boldsymbol{H}(H, 0, 0)$. This layer is heated from below such that a steady adverse temperature gradient $\beta$ $(= |\mathrm{d}T/\mathrm{d}z|)$ is maintained. Assume that both the ionized fluid and neutral gas behave like continuum fluids and that effects on the neutral component resulting from the magnetic, gravity and pressure fields are negligible. Following [2] and [5], the linearized perturbation equations governing the motion of a compressible and composite medium are

$$\varrho \frac{\partial \boldsymbol{q}}{\partial t} = -\nabla \delta p + \varrho \nu \nabla^2 \boldsymbol{q} + \boldsymbol{g}\,\delta\varrho + \frac{1}{4\pi}(\nabla \times \boldsymbol{h}) \times \boldsymbol{H} + \varrho_\mathrm{d}\, \nu_\mathrm{c}(\boldsymbol{q}_\mathrm{d} - \boldsymbol{q}), \quad (1)$$

$$\frac{\partial \boldsymbol{q}_\mathrm{d}}{\partial t} = -\nu_\mathrm{c}(\boldsymbol{q}_\mathrm{d} - \boldsymbol{q}), \quad (2)$$

$$\nabla \cdot \boldsymbol{q} = 0, \quad (3)$$

$$\frac{\partial \theta}{\partial t} = \left(\beta - \frac{g}{C_p}\right) w + \varkappa \nabla^2 \theta, \quad (4)$$

$$\nabla \cdot \boldsymbol{h} = 0, \quad (5)$$

$$\frac{\partial \boldsymbol{h}}{\partial t} = \nabla \times (\boldsymbol{q} \times \boldsymbol{H}) + \eta \nabla^2 \boldsymbol{h}. \quad (6)$$

Here $\delta p$, $\delta\varrho$, $\theta$, $\boldsymbol{q}(u, v, w)$ and $\boldsymbol{h}(h_x, h_y, h_z)$ denote, respectively, the perturbations in pressure $p$, density $\varrho$, temperature $T$, velocity $\boldsymbol{v}$ and magnetic field $\boldsymbol{H}$; $g/C_p$, $\mu$, $\nu$ $(= \mu/\varrho_m)$, $\varkappa'$, $\varkappa$ $(= \varkappa'/\varrho_m C_p)$, $\nu_\mathrm{c}$, $\boldsymbol{q}_\mathrm{d}$ and $\eta$ stand for the adiabatic gradient, the viscosity, the kinematic viscosity, the thermal conductivity, the thermal diffusivity, the collisional frequency between the two components of the composite medium, the velocity of the neutral component and the resistivity, respectively. $\alpha_m$ $(= \alpha$, say) is the coefficient of thermal expansion and $\varrho_m$ $(= \varrho$, say) is the density of the ionized medium.

Assume that the perturbation quantities are of the form

$$[w, \theta, h_z] = [W(z), \Theta(z), K(z)] \cdot \exp(i k_x x + i k_y y + n t), \quad (7)$$

where $k_x$, $k_y$ are horizontal wave numbers of the harmonic disturbance, $k^2 = k_x^2 + k_y^2$, and $n$ is the frequency.

Let $a = k d$, $\sigma = n d^2/\nu$, $p_1 = \nu/\varkappa$, $p_2 = \nu/\eta$, $\alpha_0 = \varrho_\mathrm{d}/\varrho$, $D = \mathrm{d}/\mathrm{d}z$, and let $x, y, z$ stand for the coordinates in the new unit of length $d$. Eliminating $\boldsymbol{q}_\mathrm{d}$ between (1) and (2), using Boussinesq's equation of state $\delta\varrho = -\alpha\varrho\theta$ and expression (7), (1)–(6) give

$$(D^2 - a^2)\left\{D^2 - a^2 - \sigma\left(1 + \frac{\alpha_0 \nu_\mathrm{c} d^2/\nu}{\sigma + \nu_\mathrm{c} d^2/\nu}\right)\right\} W - \left(\frac{g \alpha d^2}{\nu}\right) a^2 \Theta + \frac{i k_x H d}{4\pi\varrho\nu}(D^2 - a^2) K = 0. \quad (8)$$

$$(D^2 - a^2 - p_2 \sigma) K = -\left(\frac{H d^2}{\eta}\right) i k_x W, \quad (9)$$

$$(D^2 - a^2 - p_1 \sigma)\Theta = -\frac{d^2}{\varkappa}\left(\beta - \frac{g}{C_p}\right) W. \quad (10)$$

Eliminating $K$ and $\Theta$ between (8)–(10), we obtain

$$(D^2 - a^2)(D^2 - a^2 - p_1\sigma) \cdot \left[(D^2 - a^2 - p_2\sigma)\left\{D^2 - a^2 - \sigma\left(1 + \frac{\alpha_0 \nu_\mathrm{c} d^2/\nu}{\sigma + \nu_\mathrm{c} d^2/\nu}\right)\right\} + Q a^2 \cos^2\theta\right] W = -R\left(\frac{G-1}{G}\right) a^2 (D^2 - a^2 - p_2\sigma) W, \quad (11)$$

where $\cos\theta = k_x/k$, $G = C_p \beta/g$, $R = g\alpha\beta d^4/\nu\varkappa$ is the Rayleigh number and $Q = H^2 d^2/4\pi\varrho\nu\eta$ is the Chandrasekhar number.

Consider the case in which both the boundaries are free and the medium adjoining the fluid is non-conducting. The boundary conditions appropriate for the problem are (Chandrasekhar [1]):

$$\left.\begin{array}{l} W = D^2 W = 0, \quad \Theta = 0 \\ X = 0 \text{ and } \boldsymbol{h} \text{ is continuous} \end{array}\right\} \quad \text{at} \quad z = 0 \text{ and } 1. \quad (12)$$

In the absence of any surface current, the tangential components of the magnetic field are continuous. Hence the boundary conditions, in addition to (12), are

$$DK = 0, \quad \text{on the boundaries.} \quad (13)$$

Using the above boundary conditions, it can be shown that all the even order derivatives of $W$ must vanish for $z = 0, 1$ and hence the proper solution of $W$ characterizing the lowest mode is

$$W = W_0 \sin \pi z, \quad (14)$$

where $W_0$ is a constant. Substituting (14) in (11), we obtain the dispersion relation

$$R_1 = \left(\frac{G}{G-1}\right)\left[\frac{(1+x)\left(1+x+p_1\frac{\sigma}{\pi^2}\right)\left\{\left(1+x+p_2\frac{\sigma}{\pi^2}\right)\left(1+x+\frac{\sigma}{\pi^2}\left(1+\frac{\alpha_0\, v_c\, d^2/v}{\sigma+v_c\, d^2/v}\right)\right)\right\}+Q_1\, x\cos^2\theta}{x\left(1+x+p_2\frac{\sigma}{\pi^2}\right)}\right] \tag{15}$$

where $x = a^2/\pi^2$, $R_1 = R/\pi^4$ and $Q_1 = Q/\pi^2$.

## 3. The Stationary Convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$, and (15) reduces to

$$R_1 = \left(\frac{G}{G-1}\right)\left[\left(\frac{1+x}{x}\right)\{(1+x)^2 + Q_1\, x\cos^2\theta\}\right]. \tag{16}$$

For a fixed value of $Q_1$, let the nondimensional number $G$ accounting for the compressibility effects be also kept as fixed; then we find

$$\bar{R}_c = \left(\frac{G}{G-1}\right) R_c\,, \tag{17}$$

where $\bar{R}_c$ and $R_c$ denote, respectively, the critical Rayleigh numbers in the presence and absence of compressibility. Thus the effect of compressibility is to postpone the onset of thermal instability. Hence we obtain a stabilizing effect of compressibility. The cases $G < 1$ and $G = 1$ correspond to negative and infinite values of critical Rayleigh numbers in the presence of compressibility, which are not relevant for the present problem.

## 4. Stability of the System and Non-oscillatory Modes

Multiplying (8) by $W^*$, the complex conjugate of $W$, and using (9) and (10), we get

$$I_1 + \sigma\left(1+\frac{\alpha_0\, v_c\, d^2/v}{\sigma + v_c\, d^2/v}\right) I_2 + \frac{\eta}{4\pi\varrho v}(I_3 + p_2\,\sigma^*\, I_4) + \frac{C_p\,\alpha\,\varkappa\, a^2}{v(1-G)}(I_5 + p_1\,\sigma^*\, I_6) = 0\,, \tag{18}$$

where

$$I_1 = \int_0^1 (|D^2 W|^2 + 2a^2|DW|^2 + a^4|W|^2)\,\mathrm{d}z\,,$$

$$I_2 = \int_0^1 (|DW|^2 + a^2|W|^2)\,\mathrm{d}z\,,$$

$$I_3 = \int_0^1 (|D^2 K|^2 + 2a^2|DK|^2 + a^4|K|^2)\,\mathrm{d}z\,,$$

$$I_4 = \int_0^1 (|DK|^2 + a^2|K|^2)\,\mathrm{d}z\,,$$

$$I_5 = \int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)\,\mathrm{d}z\,,$$

$$I_6 = \int_0^1 |\Theta|^2\,\mathrm{d}z\,, \tag{19}$$

where $\sigma^*$ is the complex conjugate of $\sigma$ and the integrals $I_1 - I_6$ are all positive definite. Putting $\sigma = \sigma_\gamma + i\,\sigma_i$ and then equating the real and imaginary parts of (18), we obtain

$$I_1 + \frac{(\sigma_\gamma + v_c\, d^2/v)\,\{\sigma_\gamma^2 + \sigma_i^2 + \sigma_\gamma(1+\alpha_0)(v_c\, d^2/v)\}}{(\sigma_\gamma + v_c\, d^2/v)^2 + \sigma_i^2} I_2 + \frac{\eta}{4\pi\varrho v}(I_3 + p_2\,\sigma_\gamma I_4) + \frac{C_p\,\alpha\,\varkappa\, a^2}{v(1-G)}(I_5 + p_1\,\sigma_\gamma I_6) = 0, \tag{20}$$

and

$$i\,\sigma_i\left[\frac{\sigma_\gamma^2 + \sigma_\gamma(v_c\, d^2/v)(1+\alpha_0) + (v_c\, d^2/v)^2(1+\alpha_0) + \sigma_i^2}{(\sigma_\gamma + v_c\, d^2/v)^2 + \sigma_i^2} I_2 - \frac{\eta}{4\pi\varrho v} p_2 I_4 + \frac{C_p\,\alpha\,\varkappa\, a^2}{v(G-1)} p_1 I_6\right] = 0. \tag{21}$$

It follows from (21) that if $G > 1$, and if the magnetic field is absent, $\sigma_i = 0$. This means that the oscillatory modes are not allowed for $G > 1$ and in the absence of magnetic field. The presence of magnetic field, in contrast to nonoscillatory modes for $G > 1$ in the absence of magnetic field, introduces oscillatory modes in the system.

## 5. The Case of Overstability

Here we ask whether instability may occur as an overstability. Put $\sigma/\pi^2 = i\,\sigma_1$, it being remembered that $\sigma$ may be complex. Since for overstability, we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (15) will admit of solutions with $\sigma_1$ real. Equation (15) becomes

$$R_1 = \left(\frac{G}{G-1}\right)\left[\frac{(1+x)(1+x+i p_1 \sigma_1)\left\{(1+x+i p_2 \sigma_1)\left(1+x+i\sigma_1 \overline{1+\frac{\alpha_0 v_c d^2/v}{\sigma + v_c d^2/v}}\right) + Q_1 x \cos^2\theta\right\}}{x(1+x+i p_2 \sigma_1)}\right]. \tag{22}$$

Equating real and imaginary parts of (22) and eliminating $R_1$ between them, we get

$$(p_2^2 \pi^2 d^2/v)[p_1 \alpha_0 v_c + (1+x)(1+p_1)(\pi^2 v/d^2)]\,\sigma_1^4 + (1+x)$$
$$[(p_2 v_c d^2/v)^2 (1+p_1+\alpha_0) + (1+x)\,\pi^2 p_1 \alpha_0 (v_c d^2/v) + \pi^4 (1+p_1)(1+x)^2 + \pi^4 Q_1 x \cos^2\theta\,(p_1 - p_2)]\,\sigma_1^2$$
$$+ (1+x)[(1+x)^2(1+p_1+\alpha_0) + Q_1 x \cos^2\theta\,(p_1-p_2)](v_c d^2/v)^2 = 0. \tag{23}$$

When $p_1 \geqq p_2$, (23) is quadratic in $\sigma_1^2$ and does not allow of its roots to be such that $\mathrm{Re}(\sigma_1^2)$ is positive so that $\sigma_1$ is imaginary. Hence $p_1 \geqq p_2$, i.e.

$$\varkappa \leqq \eta \tag{24}$$

is a sufficient condition for the nonexistence of overstability. The condition (24) is the same as in the absence of compressibility as well as collisional (frictional) effects with neutrals on the thermal instability (Chandrasekhar [1]). Thus $\varkappa \leqq \eta$ is a sufficient condition for the nonexistence of overstability, which holds both in the presence or absence (Chandrasekhar [1]) of compressibility and collisional effects with neutrals on the thermal instability of a composite and compressible medium. We examine the nature of $\mathrm{d}R_1/\mathrm{d}Q_1$ to study the effect of a magnetic field on the thermal instability of a compressible and composite medium. It follows from (22) that

$$\frac{\mathrm{d}R_1}{\mathrm{d}Q_1} = \left(\frac{G}{G-1}\right) \cdot \left[\frac{(1+x)^2 + p_1 p_2 \sigma_1^2 + i\sigma_1 (1+x)(p_1-p_2)}{(1+x)^2 + p_2 \sigma_1^2}\right] \cdot (1+x)\cos^2\theta. \tag{25}$$

The imaginary part of (25) equated to zero gives

$$p_1 = p_2. \tag{26}$$

Equating the real parts of (25) and substituting (26) in it, we get

$$\frac{\mathrm{d}R_1}{\mathrm{d}Q_1} = \left(\frac{G}{G-1}\right)(1+x)\cos^2\theta. \tag{27}$$

Equation (27) implies that $\mathrm{d}R_1/\mathrm{d}Q_1$ is positive if $G > 1$. Hence for $G > 1$, the Rayleigh number increases as the magnetic field increases, showing the stabilizing effect of magnetic field.

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